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Uniform Approximation by a Non-convex Cone of Continuous Functions*

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Let Π be a collection of subsets of a compact set S in a normed linear space and K be all continuous functions f on S whose level sets, $\{s: f(s) \leqslant \alpha\}$, are in Π for all α. Then K is a cone which is not necessarily convex. The problem under consideration is to find a best uniform approximation to a continuous function on S from K. In this article, under certain conditions on Π, extremal best approximations are identified, a best approximation and its uniqueness are characterized, and Lipschitzian selections are determined. The results are illustrated by approximation problems. Analysis is also presented for the special case when K is a convex cone. Applications are given to normed vector lattices and the isotone approximation problem on order-intervals. © 1992 Academic Press, Inc.

1. INTRODUCTION

Often in analysis, sets of the form $\{s: f(s) \leqslant \alpha\}$, called the level sets of the real function f, where α is real, are used to define function classes. For example, a function is measurable if its level set is a member of a sigma-field for each α. Similarly, a function is called isotone if its level sets are members of a sigma-lattice [10, 11]. A quasi-convex function is a function whose level sets are convex [2, 13, 14]. A function is non-decreasing on $[a, b]$ if and only if its level sets are of the form $[a, c)$ or $[a, c]$, where $a \leqslant c \leqslant b$. Given a collection Π of subsets of a compact set in a normed linear space, let K (resp. K') be all the continuous (resp. bounded) real functions whose level sets are in Π. In general K (K') is a non-convex (i.e., not necessarily convex) cone. Under certain natural conditions on Π, we consider the problem of best uniform approximation of a continuous

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function by functions in K (K'). We identify extremal best approximations, characterize a best approximation and its uniqueness, and determine Lipschitzian selections. As a tool for analysis, we develop "shape preserving" transformation on sets. We illustrate the results by examples from approximation theory. We also consider the special case when K is a convex cone and give applications, among others, to normed vector lattices and an approximation problem on order-intervals.

We now define the problem in mathematical terms and introduce some notation and terminology. Let X be a normed linear space with the norm $|\cdot|$ and S be a non-empty compact subset of X. Let Π be a collection of subsets of S such that $\phi, S \in \Pi$. Let $C = C(S)$ (resp. $B = B(S)$) denote the space of continuous (resp. bounded) functions f on S with uniform norm $\|f\| = \sup\{|f(s)| : s \in S\}$. For convenience we denote the set $\{s \in S: f(s) \leqslant \alpha\}$ by $\{f \leqslant \alpha\}$. Similar notation will be used for other sets. Let K (resp. K') be all f in C (resp. B) such that $\{f \leqslant \alpha\} \in \Pi$ for all real α. We call a set M of functions on S a cone if $\lambda f \in M$ whenever $f \in M$ and $\lambda \geqslant 0$. It is easy to see that a cone M is convex if and only if $f + h \in M$ whenever $f, h \in M$. Clearly, the set K (K') defined above is a non-convex cone. Let $\Delta(f)$ (resp. $\Delta'(f)$) denote the infimum of $\|f - k\|$ for k in K (resp. K'). Given f in C, the problem is to find an f' from K (resp. K') so that $\|f - f'\|$ equals $\Delta(f)$ (resp. $\Delta'(f)$). Such an f' is called a best approximation to f from K (resp. K'). Again, f' is called an extremal or, more specifically, the maximal (resp. minimal) best approximation if $f' \geqslant g$ (resp. $f' \leqslant g$) for all best approximations g. For our problem it will be seen later that $\Delta(f) = \Delta'(f)$ when f is in C. A selection operator T which maps f in C to one of its best approximations f' in K is called a Lipschitzian selection operator (LSO) if $\|T(f) - T(h)\| \leqslant c(T)\, \|f - h\|$, for all f, h in C for some least number $c(T)$. An LSO T is an optimal LSO (OLSO) if $c(T) \leqslant c(T')$ for all LSO T' [22]. Given a set M of functions on S, define

$$\bar{f}(s) = \bar{f}_M(s) = \sup\{k(s) : k \in M, k \leqslant f\}, \qquad s \in S,$$

$$\underline{f}(s) = \underline{f}_M(s) = \inf\{k(s) : k \in M, k \geqslant f\}, \qquad s \in S.$$

If $\bar{f}$ (resp. $\underline{f}$) is in M, it is called the greatest M-minorant (resp. the smallest M-majorant) of f.

We outline briefly the contents and results of this paper. The version of the above problem for bounded functions was introduced in Section 4 of [22]. In this article, first, we develop the treatment and results for continuous functions defined on a normed linear space. Second, under certain conditions on Π, we decompose the non-convex cone K into convex cones K_x so that $K = \bigcup \{K_x : x \in S\}$ and obtain stronger results regarding the extremal best approximations, characterization of a best approximation and its uniqueness. Third, as a tool for analysis, we develop and apply nonlinear transformations on sets which essentially preserve the shape and properties of sets. In Section 2, we give algebraic and topological conditions on Π. The latter set of conditions is defined in terms of the Hausdorff metric. The set transformations are developed in Section 3. They map each of the collections of convex, star-shaped, circled, rectangular, or absolutely convex subsets of X into the same collection. In Section 7, we show that these transformations also map the set of all upper (resp. lower) subsets of an order-interval in a normed vector lattice into the same set. In Sections 4 and 5 we approximate an f in C by functions from a nonconvex cone K, and identify extremal best approximations as the shifts of the K-minorants and majorants defined earlier and isolate an LSO. The method for decomposing K is given in Section 5. A similar concept was used earlier in [18, 19] on a real interval or discrete sets for analysis of approximation problems obtaining linear time algorithms. In Section 6, we consider the special case when K is a convex cone. In each section, we present applications with examples from approximation theory. Section 7 is devoted to the analysis of normed vector lattices and the isotone approximation problem. The results for approximating an f in C by K' are similar and arise naturally during analysis of the problem. For surveys or recent work on continuous and Lipschitz continuous selections see [3–5, 22, 26]. Some related L_p-approximation problems are considered in [21].

2. Conditions on Π and Hausdorff Metric

In this section, we first present some definitions and notation and then introduce conditions on Π.

Let $D(s, r)$ and $\bar{D}(s, r)$ denote, respectively, the open and closed balls in X with center s and radius r. For $A \subset X$, let $\mathrm{cl}(A)$ denote the closure of A. If $P \subset S \subset X$, let $\mathrm{int}(P)$ denote the interior of P when regarded as a subset of S with its relative topology. That is, $s \in \mathrm{int}(P)$ if and only if there exists some $r > 0$ such that $D(s, r) \cap S \subset P$.

For $A \subset S$, define the distance function $d(s, A)$ by

$$d(s, A) = \inf\{|s - t| : t \in A\}, \qquad s \in X$$

($d(s, \phi) = \infty$). For $A \neq \phi$, we note the following. Since S is compact, there exists t in $\mathrm{cl}(A)$ such that $d(s, A) = |s - t|$. It can be easily shown that d is Lipschitzian, i.e., for all s, t in X,

$$|d(s, A) - d(t, A)| \leqslant |s - t|. \tag{2.1}$$

The Hausdorff distance σ on non-empty subsets of X is defined by

$$\sigma(E, F) = \max\{\sup\{d(s, E) : s \in F\}, \sup\{d(s, F) : s \in E\}\},$$

where $E \subset X$ and $F \subset X$ are non-empty [1, 9]. Recall that $\phi, S \in \Pi$. Let

$$\Pi_s = \{P \in \Pi : s \in P\} \cup \{\phi\}, \qquad s \in S,$$
$$\Pi' = \{S \backslash P : P \in \Pi\},$$
$$\Pi'_s = \{P' \in \Pi' : s \in P'\} \cup \{\phi\}, \qquad s \in S.$$

Then $\Pi = \bigcup \{\Pi_s : s \in S\}$ and $\Pi' = \bigcup \{\Pi'_s : s \in S\}$.

We now state conditions on Π. Not all conditions will be imposed on Π at the same time. The following are algebraic conditions.

C1. Π is closed under arbitrary intersections.

C2$'$. Π is closed under countable unions of increasing sequences of sets.

C2. If a collection of sets in Π has non-empty intersection, then the union of these sets is in Π.

C3. Π is closed under arbitrary unions.

Note that C3 implies C2 which in turn implies C2$'$. For illustrations of Π satisfying the above conditions, consider the following simple examples: C1 and C2$'$ are satisfied by convex subsets of convex S, C1 and C2 by sub-intervals of a real interval $[a, b]$, C1 and C3 by sub-intervals of the form $[a, c)$, $[a, c]$ of $[a, b]$ where $a \leqslant c \leqslant b$. Other examples are given later. The following are the topological conditions on Π.

D1. Given $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following: For all $s, t \in S$ with $|s - t| < \delta$ and for all $P' \in \Pi'_s$ with $\text{int}(S \backslash P') \neq \phi$, there exists $Q' \in \Pi'_t$ with $P' \subset Q'$ such that $\sigma(P', Q') < \varepsilon$ or, equivalently,

$$\inf_{Q'} \{\sigma(P', Q') : P' \subset Q' \in \Pi'_t\} < \varepsilon. \tag{2.2}$$

D2. Given $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following: For all $s, t \in S$ with $|s - t| < \delta$ and for all $P \in \Pi_s$, there exists $Q \in \Pi_t$ with $P \subset Q$ such that $\sigma(P, Q) < \varepsilon$ or, equivalently,

$$\inf_{Q} \{\sigma(P, Q) : P \subset Q \in \Pi_t\} < \varepsilon. \tag{2.3}$$

LEMMA 2.1. (a) *If C1 holds, then D1 is equivalent to the condition obtained by replacing* (2.2) *by* $\inf_{Q'}\{\sigma(P', Q') : Q' \in \Pi'_t\} < \varepsilon$.

(b) *If C3 holds, then D2 is equivalent to the condition obtained by replacing* (2.3) *by* $\inf_{Q}\{\sigma(P, Q) : Q \in \Pi_t\} < \varepsilon$.

Proof. We establish (a); proof for (b) is similar. Clearly D1 implies the new condition. Now suppose that the new condition holds. Then we assert that

$$\inf_{Q'} \{\sigma(P', Q') : P' \subset Q' \in \Pi'_t\} \leqslant \inf_{Q'} \{\sigma(P', P' \cup Q') : Q' \in \Pi'_t\}$$
$$\leqslant \inf_{Q'} \{\sigma(P', Q') : Q' \in \Pi'_t\}.$$

Note that C1 implies that Π' is closed under unions. Hence $P' \cup Q' \in \Pi'_t$ for all Q' in Π'_t and the first inequality follows. The second follows because $\sigma(P', P' \cup Q') \leqslant \sigma(P', Q')$ as may be easily verified. These inequalities establish the required result. The proof is complete.

PROPOSITION 2.1. (a) *Assume* C1 *and* C3 *hold. Then* D1 *implies* $\operatorname{int}(P) \in \Pi$ *whenever* $P \in \Pi$.

(b) *Assume* C1 *and* C2 *hold. Then* D2 *implies* $\operatorname{cl}(P) \in \Pi$ *whenever* $P \in \Pi$.

Proof. We first show (b). For each n let $\delta_n > 0$ be the value of δ in D2 when $\varepsilon = 1/n$. Let $P \in \Pi$. If $t \in \operatorname{cl}(P)$, then there exists $s \in P$ with $|s - t| < \delta_n$. Then $P \in \Pi_s$. By D2, we conclude that there exists $Q_{n,t} \in \Pi_t$ with $P \subset Q_{n,t}$ such that $\sigma(P, Q_{n,t}) < 1/n$. Define $Q_n = \bigcup \{Q_{n,t} : t \in \operatorname{cl}(P)\}$. Then C2 implies $Q_n \in \Pi$. It is easy to verify that $\sigma(P, Q_n) \leqslant 1/n$. Also, $\operatorname{cl}(P) \subset Q_n$ for all n since $t \in Q_{n,t}$. Let $Q = \bigcap_n Q_n$. Then $Q \in \Pi$ by C1 and $\operatorname{cl}(P) \subset Q$. Also $\sigma(P, Q) \leqslant \sigma(P, Q_n)$ for all n. Hence $\sigma(P, Q) = 0$. We conclude that $\operatorname{cl}(P) = Q \in \Pi$ and (b) is established. To show (a), let $P \in \Pi$ and $\operatorname{int}(P) \neq \phi$. Clearly C1 and C3 also hold for Π'. Then (b) applied to P' and Π' shows that $\operatorname{cl}(P') \in \Pi'$. Hence, $\operatorname{int}(P) = S \backslash \operatorname{cl}(P') \in \Pi$. The proof is complete.

3. TRANSFORMATIONS ON SETS

In this section we introduce transformation on subsets of X which essentially preserve the "shape" and properties of sets. These transformations are a tool used in analysis. The transformed sets are used to define certain conditions on Π later. These conditions are shown to apply to our examples in subsequent sections. Although we assume throughout this section that $S \subset X$ is compact, it will be seen that some results are true under weaker conditions such as closedness or even under no additional conditions.

A subset P of X is called star-shaped relative to x in P if $x + \lambda(P - x) \subset P$ for $0 \leqslant \lambda \leqslant 1$ [23]. Similarly, P is called convex if $\lambda P + (1 - \lambda) P \subset P$ for $0 \leqslant \lambda \leqslant 1$. Clearly, P is convex if and only if P is star-shaped relative to every x in P. Again, P is called balanced or circled relative to x in P if

$x+\lambda(P-x)\subset P$ for $|\lambda|\leqslant 1$ [8, 15]. Note that P is balanced relative to x if and only if P is star-shaped relative to x and $2x-P\subset P$. If P is convex and balanced relative to x, it is called absolutely convex relative to x [15].

Recall that the distance function $d(s, A)$ was defined in Section 2. For a given $P\subset S$ and $r\geqslant 0$, define

$$P_1(r)=\{s\in S\colon d(s, S\backslash P)>r\},$$

$$\bar{P}_1(r)=\{s\in S\colon d(s, S\backslash P)\geqslant r\},$$

$$P_2(r)=\{s\in S\colon d(s, P)<r\},$$

$$\bar{P}_2(r)=\{s\in S\colon d(s, P)\leqslant r\},$$

where $d(s, \phi)=\infty$. These four sets $P_i(r)$, $\bar{P}_i(r)$ are transformations of the set P. It is shown later that these transformations preserve certain properties of P. If $P\subset S$ and $Q=S\backslash P$, then, clearly, $\bar{P}_2(r)=S\backslash Q_1(r)$ and $P_2(r)=S\backslash \bar{Q}_1(r)$. Since $d(\cdot, P)$ is Lipschitz continuous by (2.1), we conclude that $P_i(r)$ is open in S and $\bar{P}_i(r)$ is closed for $i=1, 2$. We now explore the properties of these sets.

PROPOSITION 3.1. *Let $P\subset S$. Then* $\text{int}(P)=P_1(0)$ *and* $\text{cl}(P)=\bar{P}_2(0)$. *If S is convex and $r>0$ then the following holds.*

(a) $\text{int}(\bar{P}_1(r))=P_1(r)$. *Hence* $\bar{P}_1(r)=\text{cl}(P_1(r))$ *if* $\bar{P}_1(r)=\text{cl}(\text{int}(\bar{P}_1(r)))$.

(b) $\bar{P}_2(r)=\text{cl}(P_2(r))$.

Proof. The first statement is simple to prove.

(a) If $P=\phi$, the result holds. Let $P\neq\phi$. Since $P_1(r)$ is open in S and $\bar{P}_1(r)\supset P_1(r)$, we conclude that $\text{int}(\bar{P}_1(r))\supset P_1(r)$. Now suppose that $t\in\text{int}(\bar{P}_1(r))$. We show that $t\in P_1(r)$. There exists $0<\rho<r/3$ such that $\bar{D}(t, \rho)\cap S\subset \bar{P}_1(r)$. Let $E=\{u\in X\colon |u-t|=\rho\}$, $E'=E\cap S$, and $0<\varepsilon<r/3$. Then $2\rho<r-\varepsilon$ and $E'\subset\bar{P}_1(r)$. If $S=\{t\}$, then $P=S$ and, hence, $P_1(r)=S$ and $t\in P_1(r)$. Now suppose there exists s in S with $s\neq t$. Then, by convexity of S, the line segment joining s and t is in S. Hence $E'\neq\phi$ for all sufficiently small ρ. We now establish the following equalities for small ρ:

$$D(t, \rho+r-\varepsilon)\cap S=\bigcup\{D(u, r-\varepsilon)\cap S\colon u\in E\}$$
$$=\bigcup\{D(u, r-\varepsilon)\cap S\colon u\in E'\}.$$

The first equality follows from $D(t, \rho+r-\varepsilon)=\bigcup\{D(u, r-\varepsilon)\colon u\in E\}$ which holds because $\rho<r-\varepsilon$. To show the second equality, let $s\in D(u, r-\varepsilon)\cap S$ for some u in E. Then $|s-t|\leqslant|u-t|+|s-u|<\rho+r-\varepsilon$. By convexity of S, the line segment L joining t and s in S. If $|s-t|\geqslant\rho$, then there exists v in L such that $|v-t|=\rho$. Then $v\in E'$ and $|s-v|=$

$|s-t|-|v-t|<r-\varepsilon$. Hence $s\in D(v, r-\varepsilon)\cap S$. If, on the other hand, $|s-t|<\rho$, then let $x\in E'\neq\phi$ as seen above. Then we have $|s-x|\leqslant |s-t|+|t-x|<2\rho<r-\varepsilon$. Hence $s\in D(x, r-\varepsilon)\cap S$. Thus the equalities have been proved. Now by the definition of $\bar{P}_1(r)$ we have that $D(u, r-\varepsilon)\cap(S\backslash P)=\phi$ for all u in E'. Hence, by the above equalities, $D(t, \rho+r-\varepsilon)\cap(S\backslash P)=\phi$. Since ε is arbitrary, we have $d(t, S\backslash P)\geqslant \rho+r>r$ and $t\in P_1(r)$.

(b) Let $Q=S\backslash P$. Then $\bar{P}_2(r)=S\backslash Q_1(r)$ and $P_2(r)=S\backslash\bar{Q}_1(r)$. Now the required result follows by applying (a) to Q.

The proof is complete.

The following example shows that the condition of convexity on S in the above proposition cannot be dropped. Let $S=\{0, 1\}\subset R$ and $P=\{0\}$. Then $P_1(1)=\phi$, $\bar{P}_1(1)=\mathrm{int}(\bar{P}_1(1))=P$, $P_2(1)=\mathrm{cl}(P_2(1))=P$, and $\bar{P}_2(1)=S$. The following theorems show that the transformations $P_i(r)$ and $\bar{P}_i(r)$ of P preserve the "shape" and certain characteristics of P. In the proofs, we first establish the properties of the distance function d and using them establish those of $P_i(r)$ and $\bar{P}_i(r)$. It is seen in Section 7 that the transformations $P_i(r)$ and $\bar{P}_i(r)$ retain certain properties of P in a normed vector lattice. It is easy to show that if P is convex, star-shaped, or absolutely convex, then so is $\mathrm{cl}(P)$. For a proof of this fact when P is convex see [17, Theorem 2.23].

THEOREM 3.1. *Let $P\subset S$ and $P\neq\phi$. Denote $d(s, P)$ by $d(s)$. (It is Lipschitz continuous by (2.1).) Let $x\in P$.*

(a) *Assume S and P are star-shaped relative to x. Then $d(s)$ is a star-shaped function of s in S relative to x, i.e.,*

$$d(x+\lambda(s-x))\leqslant\lambda d(s), \qquad s\in S, 0\leqslant\lambda\leqslant 1. \tag{3.1}$$

If S and P are balanced relative to x, then (3.1) and the following (3.2) hold:

$$d(s)=d(2x-s), \qquad s\in S. \tag{3.2}$$

(b) *Assume S and P are convex. Then $d(s)$ is a convex function of s in S, i.e.,*

$$d(\lambda s+(1-\lambda)t)\leqslant\lambda d(s)+(1-\lambda)\,d(t), \qquad s, t\in S, 0\leqslant\lambda\leqslant 1. \tag{3.3}$$

If S and P are absolutely convex relative to x, then (3.3) (and, hence, (3.1) since $d(x)=0$) and (3.2) hold.

Consequently, the sets, $P_2(r)$ *for* $r>0$ *and* $\bar{P}_2(r)$ *for* $r \geqslant 0$, *are non-empty and star-shaped, balanced, convex, and absolutely convex (relative to* x *whenever appropriate) in the four respective cases under consideration. (Note that* $P_2(r)=\bar{P}_2(r)=S$ *for all sufficiently large* r.) *If* $r>0$, *then* $\bar{P}_2(r)=\mathrm{cl}(P_2(r))$ *holds in* (b) *and also in* (a) *if* S *is convex.*

Proof. We prove the result when S and P are star-shaped; the remaining cases are similar. Suppose $s \in S$. Then given $\varepsilon>0$, there exists $t \in P$ such that $|s-t|<d(s)+\varepsilon$. Hence, for $0<\lambda \leqslant 1$ we have $\lambda\,|s-t|<\lambda d(s)+\varepsilon$. Since P is star-shaped, $x+\lambda(t-x) \in P$ and, hence,

$$d(x+\lambda(s-x)) \leqslant |(x+\lambda(s-x))-(x+\lambda(t-x))| = \lambda\,|s-t| \leqslant \lambda d(s)+\varepsilon.$$

Thus (3.1) holds. To show that $P_2(r)$, $r>0$, is star-shaped, let $s \in P_2(r)$. Then $d(s)<r$. By (3.1) we have $d(x+\lambda(s-x)) \leqslant \lambda d(s)<r$ for $0 \leqslant \lambda \leqslant 1$. Hence, $x+\lambda(s-x) \in P_2(r)$ and $P_2(r)$ is star-shaped. Similarly, $\bar{P}_2(r)$ is star-shaped. (The convex case also appears in [23].) The last statement follows from Proposition 3.1. The proof is complete.

Since $\mathrm{cl}(P)=\bar{P}_2(0)$, by Theorem 3.1, we may conclude that $\mathrm{cl}(P)$ is respectively star-shaped, balanced, convex, and absolutely convex (relative to x whenever appropriate) if P has these attributes. If S is convex then let $\mathrm{aff}(S)$ denote the smallest affine set or linear variety containing S. Let P_0 and S_0 denote, respectively, the interior of P and S in $\mathrm{aff}(S)$. We state the following result without proof. Compare with [17, Theorem 2.23] or [22, Lemma 3.1]. Note that $\mathrm{int}(P)$ in [22] is different from $\mathrm{int}(P)$ in this article.

PROPOSITION 3.2. *Let* $P \subset S$, *where* P *and* S *are convex. If* $s \in \mathrm{int}(P)$ *and* $t \in \mathrm{cl}(P)$, *then* $\lambda s+(1-\lambda)t \in \mathrm{int}(P)$ *for* $0<\lambda \leqslant 1$. *Consequently,* $\mathrm{int}(P)$ *is convex. Furthermore, if* $\mathrm{int}(P) \neq \phi$, *then* $\mathrm{cl}(P)=\mathrm{cl}(\mathrm{int}(P))$ *and* $\mathrm{int}(P) \neq \phi$ *if and only if* $P_0 \neq \phi$.

THEOREM 3.2. (a) *Let* $P \subset S$ *where* P *and* S *are convex and* P *is open in* $\mathrm{aff}(S)$ *or* $P \subset S_0$. *Assume* $P_0 \neq \phi$. *Denote* $d(s, S\backslash P)$ *by* $e(s)$ *and* $\sup\{e(s): s \in P\}$ *by* m. *Then* $m>0$. *Also* $e(s)$ *is a concave function of* s *on* $\mathrm{cl}(P)$, *i.e.*,

$$e(\lambda s+(1-\lambda)t) \geqslant \lambda e(s)+(1-\lambda)\,e(t), \qquad s, t \in \mathrm{cl}(P),\ 0 \leqslant \lambda \leqslant 1.$$

Consequently, for $r<m$, *the sets* $P_1(r)$, $r \geqslant 0$, *and* $\bar{P}_1(r)$, $r>0$, *are non-empty convex subsets of* P *with* $\bar{P}_1(r)=\mathrm{cl}(P_1(r))$ *for* $r>0$. *(These sets are empty for all sufficiently large* r.)

(b) *Let* $S = \times\{[a_j, b_j] : 1 \leqslant j \leqslant n\}$, *with* $a_j < b_j$, *be a compact rectangle in* R^n. *Let* $P = \times\{I_j : 1 \leqslant j \leqslant n\}$ *be a rectangle in* S, *where* I_j *is a sub-interval of* $[a_j, b_j]$ *and* $\mathrm{length}(I_j) > 0$ *for all* j. *Then for* $r < \min\{\mathrm{length}(I_j) : 1 \leqslant j \leqslant n\}/2$, *the sets* $P_1(r)$, $r \geqslant 0$, *and* $\bar{P}_1(r)$, $r > 0$, *are non-empty rectangles in* P *with* $\bar{P}_1(r) = \mathrm{cl}(P_1(r))$ *for* $r > 0$.

Proof. The proof for (a) is as in [22, Sect. 3]. The last equality in (a) follows from Proposition 3.1(a), since, by Proposition 3.2, we have $\bar{P}_1(r) = \mathrm{cl}(\mathrm{int}(\bar{P}_1(r)))$ for $r > 0$. Part (b) is obvious. The proof is complete.

We now introduce two more conditions on Π.

D3. There exists $\varepsilon > 0$ such that if $P \in \Pi$ and $\mathrm{int}(P) \neq \phi$, then for every $0 < r < \varepsilon$, $\bar{P}_1(r) \in \Pi$.

D4. There exists $\varepsilon > 0$ such that if $P \in \Pi$ then for every $0 < r < \varepsilon$, $\bar{P}_2(r) \in \Pi$.

LEMMA 3.1. (a) *Condition* $D3$ *implies* $D1$.

(b) *Condition* $D4$ *implies* $D2$.

Proof. We prove (a); the proof for (b) is similar. Let $\varepsilon > 0$ and denote the epsilon in the statement of D3 by ε_0. Let $0 < r < \min\{\varepsilon, \varepsilon_0\}$. Suppose that $s, t \in S$ with $|s - t| < r$ and $P' \in \Pi'_s$ with $\mathrm{int}(S \backslash P') \neq \phi$. Then $P = S \backslash P' \in \Pi$ and $\mathrm{int}(P) \neq \phi$. Since $r < \varepsilon_0$, by D3, $\bar{P}_1(r) \in \Pi$, and if $Q' = S \backslash \bar{P}_1(r)$, then $P' \subset Q'$. Clearly $Q' \in \Pi'_t$ and $\sigma(P', Q') \leqslant r < \varepsilon$ for all s, t and P'. Thus D1 holds. The proof is complete.

LEMMA 3.2. *Assume* $C2'$ *holds. Then* $D3$ (*resp.* $D4$) *implies that for some* $\varepsilon > 0$, $P_1(r) \in \Pi$ (*resp.* $P_2(r) \in \Pi$) *for every* $0 < r < \varepsilon$ *whenever* $P \in \Pi$ (*with* $\mathrm{int}(P) \neq \phi$ *when* $D3$ *holds*). *In particular, these conclusions hold under* $C2$ *or* $C3$.

Proof. With $\varepsilon > 0$ as in D3, let $\delta_n = r + (\varepsilon - r)/(2n)$. Then $r < \delta_{n+1} < \delta_n < \varepsilon$ and $\delta_n \to r$. Hence, $P_1(r) = \bigcup_n \bar{P}_1(\delta_n)$ and $P_1(r) \in \Pi$. The proof for D4 is similar. The proof is complete.

Conditions D3 and D4 may appear to be too strong, but they apply to our examples and, hence, are sufficient for our purpose. Nevertheless, more general workable conditions may also be introduced.

4. BEST APPROXIMATION FROM A NON-CONVEX CONE K

In this section we identify a best approximation to f in C and an LSO when K is non-convex. The corresponding problem for f in B and cone

$K' \subset B$ was introduced in Example 4.3 of [22]. The preliminaries presented in the previous sections enable us to develop the continuous case here.

For $A \subset S$, we let $\mathrm{pi}(A) = \bigcap \{P \in \Pi : A \subset P\}$. If C1 holds, then clearly $\mathrm{pi}(A) \in \Pi$. We collect some properties of K in the following lemma.

LEMMA 4.1. (a) *If C1 holds, then K (K') is a closed cone.*

(b) *If C1 and C3 hold, then K (K') is a closed convex cone.*

(c) *If $k \in K$ (resp. K'), then $k + \alpha \in K$ (resp. K') for all real α.*

(d) *If C1 and C2′ hold, then $k \in K$ (K') if and only if $\{k < \alpha\} \in \Pi$ for all α.*

Proof. The proof involves some elementary arguments and an application of the following inequalities for functions k_n, k, and h. If $\|k_n - k\| = \delta_n \to 0$, then $\{k \leqslant \alpha\} = \bigcap_n \{k_n \leqslant \alpha + \delta_n\}$. Also, $\{k + h \leqslant \alpha\} = \bigcup_\beta \{\{k \leqslant \beta\} \cap \{h \leqslant \alpha - \beta\}\}$, $\{k < \alpha\} = \bigcup_n \{k \leqslant \alpha - 1/n\}$, and $\{k \leqslant \alpha\} = \bigcap_n \{k < \alpha + 1/n\}$. The proof is complete.

PROPOSITION 4.1. (a) *Assume C1 and D1 hold. Let $f \in C$ and define*

$$f^0(P) = \inf\{f(t) : t \in S \backslash P\}, \qquad P \in \Pi, \tag{4.1}$$

$$\bar{f}(s) = \sup\{f^0(P) : P \in \Pi, s \in S \backslash P\}, \qquad s \in S. \tag{4.2}$$

Then $\bar{f} \in C$, and it is the greatest K-minorant of f. (It is also the greatest K'-minorant of f.)

(b) *Assume C1, C2′, and D1 hold. Then h in K is the greatest K-minorant of f in C if and only if*

$$\{h < \alpha\} = \mathrm{pi}\{f < \alpha\} \qquad \text{for all real } \alpha. \tag{4.3}$$

Proof. (a) Using C1 and a proof as in Proposition 4.3 of [22], we may show that $\bar{f} \in K'$ and is the greatest K'-minorant of f. (Note that the framework of [22] is slightly different from that of this article.) Hence if we show that $\bar{f} \in C$, it will follow that $\bar{f}$ is the greatest K-minorant of f. To show continuity, let $\varepsilon > 0$. Then there exists $\rho > 0$ such that $|f(u) - f(v)| < \varepsilon/2$ whenever $u, v \in S$ and $|u - v| < \rho$. Again, by D1, there exists $\delta > 0$ such that if $|s - t| < \delta$ and $P' \in \Pi'_s$ with $\mathrm{int}(S \backslash P') \neq \phi$, then for some $Q' \in \Pi'_t$ with $P' \subset Q'$ we have $\sigma(P', Q') < \rho$. Now suppose that $s, t \in S$ and $|s - t| < \delta$. Then by (4.2), there exists $P \in \Pi$ with $s \in P' = S \backslash P$ such that $\bar{f}(s) \leqslant f^0(P) + \varepsilon/2$. Note that $P' \in \Pi'_s$. First assume that $\mathrm{int}(P) \neq \phi$. Then, we can find $Q' \in \Pi'_t$ with $P' \subset Q'$ and $\sigma(P', Q') < \rho$. Let $Q = S \backslash Q'$. Then $Q \in \Pi$ and $t \in S \backslash Q$. Hence $\bar{f}(t) \geqslant f^0(Q)$. If $u \in Q' \backslash P' = P \backslash Q$, then by the definition of σ, there exists $v \in P' = S \backslash P$ such that $|u - v| < \rho$. We then have $f(u) \geqslant f(v) - \varepsilon/2$. It follows that $f^0(Q) \geqslant f^0(P) - \varepsilon/2$. Hence $\bar{f}(s) - \bar{f}(t) \leqslant f^0(P) - f^0(Q) + \varepsilon/2 \leqslant \varepsilon$. If $\mathrm{int}(P) = \phi$, then $f^0(P) = \inf(f) = \theta$, say. Hence

$\bar{f}(s) \leqslant \theta + \varepsilon/2$. Also for all $t \in S$ we have $\bar{f}(t) \geqslant \theta$ as may be easily verified. Consequently, $\bar{f}(s) - \bar{f}(t) \leqslant \varepsilon$. A symmetric argument establishes that $|\bar{f}(s) - \bar{f}(t)| \leqslant \varepsilon$. Thus $\bar{f}$ is continuous.

(b) By Lemma 4.1(d), $\{h < \alpha\} \in \Pi$ for all α. Suppose $h = \bar{f}$ is the greatest K-minorant; then $h \leqslant f$ and $\{h < \alpha\} \supset \{f < \alpha\}$. Since $\{h < \alpha\}$ is in Π, we have $\{h < \alpha\} \supset \mathrm{pi}\{f < \alpha\} = P$, say. Suppose $s \in S$ and $h(s) < \alpha$. If $s \in S \backslash P$, then $f^0(P) \leqslant h(s) < \alpha$. Hence there exists $t \in S \backslash P$ with $f(t) < \alpha$, a contradiction to the definition of P. Hence $s \in P$ and (4.3) holds. Conversely, if (4.3) holds for some h in K, then $\{h < \alpha\} \supset \{f < \alpha\}$ for all α and hence $h \leqslant f$. If $k \in K$ and $k \leqslant f$, then $\{k \leqslant \alpha\} \supset \{f \leqslant \alpha\}$ and hence $\{k \leqslant \alpha\} \supset \mathrm{pi}\{f \leqslant \alpha\}$. We conclude by (4.3) that $\{k \leqslant \alpha\} \supset \{h \leqslant \alpha\}$ for all α and hence $k \leqslant h$. The proof is complete.

We remark that a result similar to Proposition 4.1(b) may also be established for Example 4.3 of [22]. Recall the definitions of the maximal and minimal best approximations from Section 1.

THEOREM 4.1. *Assume $C1$ and $D1$ hold. Let $f \in C$ and $\bar{f}$ in C be the greatest K-minorant of f. Then $\Delta'(f) = \Delta(f) = \|f - \bar{f}\|/2$. Also $f' = \bar{f} + \Delta(f)$ is in C and is the maximal best approximation to f from K or K'. Furthermore, $\|f' - h'\| \leqslant 2\,\|f - h\|$ for all $f, h \in C$. The operator $T\colon C \to K$ defined by $T(f) = f'$ is an LSO with $c(T) = 2$. If a best approximation is unique then it equals $\bar{f} + \Delta(f)$.*

Proof. By Proposition 4.3 of [22] and subsequent discussion there, we have $\Delta'(f) = \|f - \bar{f}\|/2$ and $f' = \bar{f} + \Delta'(f)$ is the maximal best approximation to f from K'. By Proposition 4.1, $\bar{f}$ is continuous. It follows that f' is also the maximal best approximation from K and $\Delta'(f) = \Delta(f)$. By Lemma 4.1(c) and Proposition 4.1(a), K satisfies the first two conditions stated in Section 1 of [22]. Hence, the next two assertions of the theorem concerning LSO follow from Theorem 2.1(a) of [22]. (We conclude $c(T) = 2$ by the example on $S = [0, 3]$ given in [19, p. 78], since the framework of this section applies to that example.) The last assertion concerning uniqueness follows because the unique best approximation must equal the maximal best approximation. The proof is complete.

Immediately below we illustrate applications of the above results to approximation problems. We use the set transformations developed in Section 3.

EXAMPLE 4.1. Approximation by continuous functions with star-shaped (resp. balanced) level sets.

Let $S \subset X$ be compact and star-shaped (resp. balanced) relative to some x in S. Let Ω consist of all subsets of S which are star-shaped (resp. balanced) relative to x including ϕ and S. Let K be the set of all k in C

such that $\{k \leqslant \alpha\} \in \Omega$ for all α. We find a best approximation to f in C from K.

We first transform the problem to our earlier framework by defining Π. Indeed, let $\Pi = \{P: S \backslash P \in \Omega\}$. Clearly, Ω satisfies C1 and C3, and hence so does Π. By Lemma 4.1(b), as applied to Ω, we conclude that K is a closed convex cone. Since C3 implies C2′, Lemma 4.1(d) shows us that $k \in K$ if and only if $\{k < \alpha\} \in \Pi$ for all α or, equivalently, $\{k \geqslant \alpha\} \in \Pi$ for all α.

LEMMA 4.2. *Condition D3 applies to Π.*

Proof. If $P \in \Pi$, $P \neq S$, and $\operatorname{int}(P) \neq \phi$ then $Q = S \backslash P$ is non-empty and star-shaped (resp. balanced). Hence, by Theorem 3.1, $Q_2(r)$ is non-empty and star-shaped (resp. balanced) for all $r > 0$. Since $\bar{P}_1(r) = S \backslash Q_2(r)$ as may be easily verified, we have that $\bar{P}_1(r) \in \Pi$ for all $r > 0$. If $P = S$ then $\bar{P}_1(r) = S \in \Pi$. The proof is complete.

We conclude by Lemma 3.1 that D1 applies to Π. Note that D1 does not apply to Ω and hence Π was introduced. Since K is defined by the sets of the form $\{k \geqslant \alpha\}$ instead of $\{k \leqslant \alpha\}$, symmetric versions of Proposition 4.1 and Theorem 4.1 apply. In particular, we replace $\bar{f}$ there by $\underline{f}$ which is the smallest K-majorant of f, inf (resp. sup) by sup (resp. inf), let $f' = \underline{f} - \Delta(f)$, and reverse the strict inequalities in (4.3).

We revert to this problem in Section 6 and show that the application of condition C3 does not strengthen the results.

EXAMPLE 4.2. Approximation by continuous functions with rectangular level sets.

Let $S = \times\{[a_j, b_j]: 1 \leqslant j \leqslant n\}$, where $a_j < b_j$, be a compact rectangle in R^n. Let Π consist of all rectangles contained in S including ϕ and S. Clearly C1 and C2′ apply and K is a closed cone. Again, if $P \in \Pi$ and $\operatorname{int}(P) \neq \phi$, where $P = \times\{I_j: 1 \leqslant j \leqslant n\}$, then $\operatorname{length}(I_j) > 0$ for all j, as may be easily seen. Hence, by Theorem 3.2(b), $\bar{P}_1(r)$ is a non-empty rectangle for sufficiently small r. Thus D3 and hence D1 hold. We conclude that Proposition 4.1 and Theorem 4.1 apply.

We remark that the framework of this section, and hence Proposition 4.1 and Theorem 4.1 also apply to the problem of approximation by continuous quasi-convex functions on R^n considered in [22] extended to a vector space.

5. DECOMPOSITION OF A NON-CONVEX CONE K AND CHARACTERIZATION OF A BEST APPROXIMATION

In this section, under certain conditions on the non-convex cone K, we decompose it into convex cones K_x, $x \in S$ so that $K = \bigcup K_x$, and use this decomposition to characterize a best approximation and its uniqueness.

Recall the definition of Π_x from Section 2. Define

$$K_x = \{k \in C \colon \{k \leqslant \alpha\} \in \Pi_x \text{ for all real } \alpha\}, \qquad x \in S,$$

$$K'_x = \{k \in B \colon \{k \leqslant \alpha\} \in \Pi_x \text{ for all real } \alpha\}, \qquad x \in S.$$

The following lemma is immediate; proof of part (c) is similar to that of Lemma 4.1(b) and (d).

LEMMA 5.1. (a) *K_x(resp. K'_x) is the set of all k in K (resp. K') such that $k(x) = \min\{k(s) \colon s \in S\}$. Also, $K = \bigcup \{K_x \colon x \in S\}$.*

(b) *If C1 and C2 hold, then Π_x is closed under arbitrary unions and intersections.*

(c) *If C1 and C2 hold, then K_x (resp. K'_x) is a closed convex cone. Furthermore, $k \in K_x$ (resp. K'_x) if and only if $\{k < \alpha\} \in \Pi_x$ for all α.*

(d) *If $k \in K_x$ (resp. K'_x), then $k + \alpha \in K_x$ (resp. K'_x) for all α.*

Lemma 5.1(a) gives the promised decomposition of K. We point out that, in general, $K' \neq \bigcup \{K'_x \colon x \in S\}$. For f in B, we let

$$\Delta_x = \Delta_x(f) = \inf\{\|f - k\| \colon k \in K_x\},$$

$$\Delta'_x = \Delta'_x(f) = \inf\{\|f - k\| \colon k \in K'_x\}.$$

Clearly, $\Delta'_x \leqslant \Delta_x$ and $\Delta(f) = \inf\{\Delta_x \colon x \in S\}$ since $K_x \subset K'_x$ and $K = \bigcup K_x$. Define

$$U_{x,s} = \bigcap \{P \in \Pi_x \colon s \in P\},$$

$$V_{x,s} = \bigcup \{P \in \Pi_x \colon s \in S \backslash P\}, \qquad s \neq x,$$

$$= \phi, \qquad \text{otherwise.}$$

Condition C1 (resp. C2) implies that $U_{x,s}$ (resp. $V_{x,s}$) is in Π_x. Clearly $U_{x,s} = U_{s,x}$. For f in C, let

$$Y = \{y \in S \colon f(y) = \min\{f(s)\}\},$$

$$Z = \bigcap \{P \in \Pi \colon Y \subset P\},$$

$$Y^* = \{s \in S \colon f(s) \leqslant \min\{f(s)\} + 2\Delta(f)\},$$

$$Z^* = \bigcup \{P \in \Pi \colon Z \subset P \subset Y^*\}.$$

Since $f \in C$, we have $Y \neq \phi$. Condition C1 (resp. C2) implies that Z (resp. Z^*) is in Π. It will be seen later that $Z^* \neq \phi$.

PROPOSITION 5.1. *Assume C1 and C2 hold. Let $f \in B$ and define*

$$\bar{f}_x(s) = \inf\{f(t) : t \in S \backslash V_{x,s}\}, \qquad s \in S, \tag{5.1}$$

$$\underline{f}_x(s) = \sup\{f(t) : t \in U_{x,s}\}, \qquad s \in S. \tag{5.2}$$

Then $\bar{f}_x$ and $\underline{f}_x$ are in K'_x and are, respectively, the greatest K'_x-minorant and the smallest K'_x-majorant of f with $\Delta'_x = \|f - \bar{f}_x\|/2 = \|f - \underline{f}_x\|/2 = \|\bar{f}_x - \underline{f}_x\|/2$. Also, $\bar{f}_x + \Delta'_x$ and $\underline{f}_x - \Delta'_x$ are, respectively, the maximal and minimal best approximation to f from K'_x. A g in K'_x is a best approximation to f if and only if $\underline{f}_x - \Delta'_x \leqslant g \leqslant \bar{f}_x + \Delta'_x$. Furthermore, g is unique if and only if $\underline{f}_x - \bar{f}_x = \delta$ for some $\delta \geqslant 0$ in which case $\delta = 2\Delta'_x$ and $g = \underline{f}_x - \Delta'_x = \bar{f}_x + \Delta'_x$. (Both $\bar{f}_x$ and $\underline{f}_x$ may be characterized by statements similar to Proposition 4.1(b).)

Proof. Lemma 5.1(b) shows that Proposition 4.4 of [22] is applicable to K'_x. We conclude that $\bar{f}_x$ and $\underline{f}_x$ are respectively the minorant and majorant as stated. These observations and Lemma 5.1(d) show that K'_x satisfies all the three conditions stated in Section 1 of [22]. Then Theorem 2.1(c) of [22] gives the required results. The uniqueness statement is established in the remarks following Theorem 2.3 of [22]. The proof is complete.

In what follows, we let $\inf\{f(s) : s \in S\} = \theta$ for convenience.

LEMMA 5.2. $\bar{f} \geqslant \theta, \bar{f}_x \geqslant \theta = \bar{f}_x(x)$ *for all* $x \in S$.

Proof. These results follow at once from (4.1), (4.2), and (5.1). The proof is complete.

LEMMA 5.3. *Assume C1, C2, and D1 hold and $f \in C$. Then*

$$Y \subset Z \subset \text{cl}(Z) \subset \{\bar{f} = \min(f)\} \subset Z^* \subset Y^*.$$

Proof. The conditions imply that $Z, Z^* \in \Pi$ and $\bar{f}$ is continuous by Proposition 4.1. If $y \in Y$ then $f(y) = \theta$. Since by Lemma 5.2, $\theta \leqslant \bar{f} \leqslant f$, we have that $\bar{f}(y) = \theta$. Hence $Y \subset \{\bar{f} = \theta\} = P$, say. Since $P \in \Pi$, we conclude that $Y \subset Z \subset P$. By continuity of $\bar{f}$, P is closed and $\text{cl}(Z) \subset P$. Now let $s \in S$ and $\bar{f}(s) = \theta$. Since $\|f - \bar{f}\| = 2\Delta(f)$, we have $f(s) \leqslant \bar{f}(s) + 2\Delta(f) = \theta + 2\Delta(f)$. Hence $s \in Y^*$ and $P \subset Y^*$. But $Z \subset P \in \Pi$ and hence $P \subset Z^* \subset Y^*$. The proof is complete.

PROPOSITION 5.2. *Assume C1, C2, D1, D2 hold for Π. Let $f \in C$.*

(a) *For each z in Z, $\bar{f}_z$ is continuous and $\bar{f}_z = \bar{f}$. It is the greatest K_z-minorant of f for all z in Z. For each x in $S \backslash Z$, $\bar{f} \geqslant \bar{f}_x$ and $\bar{f}_x$ is the greatest K'_x-minorant of f.*

(b) *The set* $\{\underline{f}_x : x \in S\}$ *is equi-continuous. For each* x *in* S, $\underline{f}_x$ *is the smallest* K_x*-majorant of* f. *Furthermore,* $\underline{f}_x$ *is a continuous function of* x *in the norm topology for* C; *i.e., if* $y \to x$, *then* $\|\underline{f}_y - \underline{f}_x\| \to 0$.

(c) *For all* x, y *in* S, *the following holds*: $\Delta'_x = \Delta_x = \|\bar{f}_x - \underline{f}_x\|/2$, $|\Delta_x - \Delta_y| \leqslant \|\bar{f}_x - \bar{f}_y\|/2$, *and* $|\Delta_x - \Delta_y| \leqslant \|\underline{f}_x - \underline{f}_y\|/2$. *Also* Δ_x *is a continuous function of* x.

Proof. We use Lemma 5.2, the notation (4.1) and $\theta = \min(f)$ frequently in this proof.

(a) Let $s, x \in S$. By C2, $V_{x,s} \in \Pi_x$. Since $s \in S \backslash V_{x,s}$, by (4.2) and (5.1) we have $\bar{f}(s) \geqslant f^0(V_{x,s}) = \bar{f}_x(s)$, if $s \neq x$. By Lemma 5.2, we conclude that $\bar{f}(x) \geqslant \bar{f}_x(x)$. Thus $\bar{f} \geqslant \bar{f}_x$. In particular, $\bar{f} \geqslant \bar{f}_z$ for z in Z.

We now show that $\bar{f} = \bar{f}_z$ for z in Z. Let $z \in Z$. As shown above $\bar{f}(s) \geqslant \bar{f}_z(s) \geqslant \theta$ for all s in S. Hence if $s \in S$ and $\bar{f}(s) = \theta$, then $\bar{f}(s) = \bar{f}_z(s) = \theta$. Now suppose that $s \in S$ and $\bar{f}(s) > \theta$. Define $Q = \{t \in S : \bar{f}(t) < \bar{f}(s)\}$. Then, by Lemma 4.1(d), $Q \in \Pi$. Also by Lemma 5.3, $Z \subset Q$. Again since $s \in S \backslash Q$ and $z \in Q$, by the definition of $V_{z,s}$, we have $Q \subset V_{z,s}$. We conclude that $\bar{f}_z(s) \geqslant \inf\{f(t) : t \in S \backslash Q\} = f^0(Q)$. Now if $t \in S \backslash Q$, then, by the definition of Q, we have $f(t) \geqslant \bar{f}(t) \geqslant \bar{f}(s)$. Hence $f^0(Q) \geqslant \bar{f}(s)$ and $\bar{f}_z(s) \geqslant \bar{f}(s)$. We have now shown that $\bar{f}_z = \bar{f}$. Since, by Proposition 4.1, $\bar{f}$ is continuous, so is $\bar{f}_z$. Now, by Proposition 5.1, $\bar{f}_z$ is the greatest K'_z-minorant of f and it is continuous. Hence $\bar{f}_z$ is the greatest K_z-minorant of f. Again, by Proposition 5.1, $\bar{f}_x$, $x \in S \backslash Z$, is the minorant as stated.

(b) Given $\varepsilon > 0$ there exists $\rho > 0$ such that $|f(u) - f(v)| < \varepsilon$ whenever $|u - v| < \rho$. By D2, there exists $\delta > 0$ such that if $|s - t| < \delta$ and $P \in \Pi_s$, then for some $Q \in \Pi_t$ with $P \subset Q$ we have $\sigma(P, Q) < \rho$. Now let $x, s, t \in S$ with $|s - t| < \delta$. Then since $P = U_{x,s} \in \Pi_s$, there exists $Q \in \Pi_t$ with $P \subset Q$ and $\sigma(P, Q) < \rho$. But since $x \in Q$, $U_{x,t} \subset Q$. If $u \in Q \backslash P$, there exists $v \in P$ such that $|u - v| < \rho$ and, hence, $|f(u) - f(v)| < \varepsilon$. We then have by (5.2)

$$\underline{f}_x(t) \leqslant \sup\{f(u) : u \in Q\} \leqslant \sup\{f(v) : v \in P\} + \varepsilon = \underline{f}_x(s) + \varepsilon,$$

for all x. A symmetric argument completes the proof of $|\underline{f}_x(t) - \underline{f}_x(s)| \leqslant \varepsilon$. Thus $\{\underline{f}_x\}$ is equi-continuous. By Proposition 5.1, $\underline{f}_x$ is the smallest K'_x-majorant of f and it is continuous. Hence it is the smallest K_x-majorant of f. Now since $U_{x,s} = U_{s,x}$, we have $\underline{f}_x(s) = \underline{f}_s(x)$ for all x, s. Hence equi-continuity of $\{\underline{f}_x\}$ implies that $\underline{f}_x$ is a continuous function of x.

(c) By Proposition 5.1, $\Delta'_x = \|f - \underline{f}_x\|/2 = \|\bar{f}_x - \underline{f}_x\|/2$ and $\underline{f}_x - \Delta'_x$ is the minimal best approximation to f from K'. Since $\underline{f}_x$ is continuous by (b), we conclude that $\Delta'_x = \Delta_x$. Since $\Delta_x = \|f - \bar{f}_x\|/2$ and $\Delta_y = \|f - \bar{f}_y\|/2$, using the triangle inequality, we have $|\Delta_x - \Delta_y| \leqslant \|\bar{f}_x - \bar{f}_y\|/2$. The proof of the last inequality is similar. The continuity of Δ_x follows from (b). The proof is complete.

We now state our main results of this section. We let $\Delta' = \Delta'(f)$ and $\Delta = \Delta(f)$.

THEOREM 5.1. *Assume conditions $C1$, $C2$, $D1$, and $D2$ hold. Let $f \in C$.*

(a) Extremal best approximations and errors. *For all z in Z^*,*

$$\Delta' = \Delta = (1/2)\,\|f - \bar{f}\| = (1/2)\,\|f - \underline{f}_z\| = (1/2)\,\|\bar{f} - \underline{f}_z\|. \tag{5.3}$$

$\bar{f} + \Delta$ is the maximal best approximation to f from K. Each $\underline{f}_z - \Delta$ for z in Z^* is also a best approximation to f from K; in fact, it is the minimal best approximation from K_z, and $\underline{f}_z - \Delta \leqslant \bar{f} + \Delta$. The above conclusions hold when K and K_z are replaced by K' and K'_z respectively.

(b) Characterization of best approximations. *A g in K (resp. K') is a best approximation to f from K (resp. K') if and only if there exists z in Z^* such that $\underline{f}_z - \Delta \leqslant g \leqslant \bar{f} + \Delta$.*

COROLLARY. *$\Delta \leqslant \Delta_z = \Delta'_z$ with equality holding if and only if $z \in Z^*$. Furthermore, Z^* is compact.*

Proof. (a) We first establish the results for K and K_z. By Theorem 4.1, $\bar{f}$ is in K, $\Delta = (1/2)\,\|f - \bar{f}\|$, and $\bar{f} + \Delta$ is the maximal best approximation. Suppose first that $z \in Z$. Then, by Proposition 5.2(a) and (c), we have $\bar{f} = \bar{f}_z$ and $\Delta'_z = \Delta_z$. Hence, by Proposition 5.1, we find that

$$\Delta = (1/2)\,\|f - \bar{f}\| = (1/2)\,\|f - \bar{f}_z\| = (1/2)\,\|f - \underline{f}_z\| = \Delta'_z = \Delta_z.$$

Since $\underline{f}_z$ is continuous by Proposition 5.2(b), we conclude using the above equalities and Proposition 5.1 again that $\underline{f}_z - \Delta$ is a best approximation and is, indeed, the minimal best approximation from K_z. Furthermore, $\underline{f}_z - \Delta \leqslant \bar{f}_z + \Delta = \bar{f} + \Delta$.

Now suppose that $z \in Z^* \backslash Z$. We first show that $\|f - \bar{f}_z\| = 2\Delta$. If $s \in Y^*$, then $f(s) - 2\Delta \leqslant \min\{f(s)\} = \theta$. Also, $\bar{f}_z(s) \geqslant \theta$. Hence $0 \leqslant f(s) - \bar{f}_z(s) \leqslant 2\Delta$. Now suppose $s \in S \backslash Y^*$. We assert that $\bar{f}_z(s) = \bar{f}(s)$. Let $P \in \Pi$ with $s \in S \backslash P$. If $Y \cap (S \backslash P) \neq \phi$, then by (4.1) we have $f^0(P) = \theta$ and, hence, $\bar{f}_z(s) \geqslant \theta = f^0(P)$. If $Y \cap (S \backslash P) = \phi$ then $Y \subset P$. But, by Lemma 5.3, $Y \subset Z^*$. Hence, by $C2$, we have $Q = P \cup Z^* \in \Pi$. Since $s \in S \backslash Y^* \subset S \backslash Z^*$ we have that $s \in S \backslash Q$. Also $z \in Q$ and hence $Q \subset V_{z,s}$. Then

$$\bar{f}_z(s) = f^0(V_{z,s}) \geqslant f^0(Q) \geqslant f^0(P).$$

Thus $\bar{f}_z(s) \geqslant f^0(P)$ for all P in Π with $s \in S \backslash P$. Hence $\bar{f}_z(s) \geqslant \bar{f}(s)$ and, by Proposition 5.2(a), $\bar{f}_z(s) = \bar{f}(s)$. It follows that $0 \leqslant f(s) - \bar{f}_z(s) = f(s) - \bar{f}(s) \leqslant 2\Delta$. We have thus shown that $\|f - \bar{f}_z\| \leqslant 2\Delta$. Now $\Delta \leqslant \Delta_z = \Delta'_z$ by

Proposition 5.2(c). By Proposition 5.1, we have $2\Delta'_z = \|f - \underline{f}_z\| = \|f - \bar{f}_z\| \leqslant 2\Delta$. Hence $\|f - \underline{f}_z\| = 2\Delta$. Again, by Proposition 5.2(b), $\underline{f}_z$ is continuous and we conclude that $\underline{f}_z - \Delta$ is a best approximation. By Proposition 5.1, it is the minimal best approximation from K_z. Now, Propositions 5.1 and 5.2(a) show that $\underline{f}_z - \Delta \leqslant \bar{f}_z + \Delta \leqslant \bar{f} + \Delta$.

We have established above that for all z in Z^*, $\Delta = \|f - \underline{f}_z\|/2$ and $\underline{f}_z - \Delta \leqslant \bar{f} + \Delta$. Since $\underline{f}_z \geqslant f \geqslant \bar{f}$ this gives $2\Delta \geqslant \underline{f}_z - \bar{f} \geqslant \underline{f}_z - f \geqslant 0$. We conclude that $\|\bar{f} - \underline{f}_z\| = 2\Delta$ and (5.3) is proved.

The proof for K' and K'_z is similar to the above.

(b) Suppose g in K or K' satisfies $\underline{f}_z - \Delta \leqslant g \leqslant \bar{f} + \Delta$. Then, since $\underline{f}_z - \Delta$ and $\bar{f} + \Delta$ are best approximations, we conclude that g is a best approximation. Conversely, suppose first that g in K is a best approximation. Then, since $K = \bigcup K_x$, we have that $g \in K_z$ for some z in S. By Lema 5.1(a), $g(z) \leqslant g(s)$ for all s in S. Also $f(s) - \Delta \leqslant g(s) \leqslant f(s) + \Delta$ for all s in S. Hence if $y \in Y$, then $f(y) = \theta$ and $f(z) - \Delta \leqslant g(z) \leqslant g(y) \leqslant f(y) + \Delta$. We conclude that $f(z) \leqslant \theta + 2\Delta$, i.e., $z \in Y^*$.

To show $z \in Z^*$ assume to the contrary that $z \in Y^* \backslash Z^*$. We assert that there exists $t \in S \backslash Y^*$ such that $Y \cap (S \backslash V_{z,t}) \neq \phi$. To the contrary suppose that for all s in $S \backslash Y^*$ we have $Y \subset V_{z,s}$. Then since $V_{z,s} \in \Pi$ we have $Z \subset V_{z,s}$. Let $A = \bigcap \{V_{z,s} : s \in S \backslash Y^*\}$. Since $s \in S \backslash V_{z,s}$, we have $Z \subset A \subset Y^*$. Again, since $A \in \Pi$ by condition C1, we find that $Z \subset A \subset Z^*$. This is a contradiction, because for all s in $S \backslash Y^*, z \in V_{z,s}$ and hence $z \in A$. We have thus established the assertion made above for t. It follows by (5.1) that $\bar{f}_z(t) = f^0(V_{z,t}) = \theta$. Since $t \in S \backslash Y^*$ we have $f(t) > \theta + 2\Delta$. Thus $f(t) - \bar{f}_z(t) > 2\Delta$. Now Proposition 5.1 shows that $\|f - g\| \geqslant \Delta'_z = (1/2)\|f - \bar{f}_z\| > \Delta$. Thus g is not a best approximation, a contradiction. We conclude that $z \in Z^*$. Now (a) shows that $\underline{f}_z - \Delta \leqslant g \leqslant \bar{f} + \Delta$. The above arguments also establish the first statement of the corollary. The second statement follows from the first, since by Proposition 5.2(c), Δ_x is a continuous function of x on compact S.

Now let g in K' be a best approximation. Then $\|f - g\| = \Delta$ and $g \leqslant \bar{f} + \Delta$ by Theorem 4.1. Let $\tilde{g}$ denote the lower semi-continuous envelope of g defined by

$$\tilde{g}(s) = \min\{g(s), \liminf\{g(t) : t \to s\}\}.$$

It is easy to verify that $\{\tilde{g} \leqslant \alpha\} = \bigcap_n \operatorname{cl}\{g \leqslant \alpha + 1/n\}$ for all α. By Proposition 2.1(b), Π is closed under the closure operation on sets. Since $\{g \leqslant \alpha + 1/n\}$ is in Π, its closure is in Π. Now C1 ensures that $\{\tilde{g} \leqslant \alpha\}$ is in Π. Thus $\tilde{g} \in K'$. Since $\tilde{g}$ is lower semi-continuous, a minimizer z of $\tilde{g}$ exists. Then $\tilde{g} \in K'_z$. Also, by continuity of f, we have $\|f - \tilde{g}\| = \|f - g\| = \Delta = \Delta'_z$. Thus $\tilde{g}$ is a best approximation from K' and K'_z. The corollary now shows that $z \in Z^*$. By (a), we have $\underline{f} - \Delta \leqslant \tilde{g} \leqslant g$. The proof is complete.

If we define subsets in Π using parameters, then Z^* may be determined by adjusting the values of the parameters so that it is the largest set in Π contained in Y^*. This is illustrated in Examples 5.1 and 5.2. Note that $U_{s,s}$ is the smallest set in Π containing s.

THEOREM 5.2. Uniqueness of best approximations. *Assume conditions $C1$, $C2$, $D1$, and $D2$ hold. Let $f \in C$ and g denote a best approximation to f from K (resp. K'). Then the following* (a)–(c) *hold.*

(a) *The following three statements are equivalent.*

(i) *g is unique.*

(ii) *$\underline{f}_z - \bar{f} = \delta$ for all z in Z^* and some $\delta \geqslant 0$. (In this case $\delta = 2\Delta$ and $g = \bar{f} + \Delta = \underline{f}_z - \Delta$ for all z in Z^*.)*

(iii) *$\bar{f}_z = \bar{f}$ for all z in Z^* and a best approximation from K_z (resp. K'_z) is unique for each z in Z^*. (In this case, g is also the unique best approximation from K_z (resp. K'_z).)*

(b) *Suppose that whenever $P \in \Pi$, P is closed, and $s \in S \backslash P$, there exists $Q \in \Pi$ such that Q is open in S, $P \subset Q$, and $s \in S \backslash Q$. Then g is unique if and only if* $\mathrm{cl}(Z) = Z^*$ *and a best approximation from K_z (resp. K'_z) is unique for each z in Z^*. (In this case, g is also the unique best approximation from K_z (resp. K'_z).)*

(c) *In the following, statements* (i)–(v) *are equivalent and* (vi) *implies* (i).

(i) *g is unique if and only if $f \in K$ (resp. K'), and then $g = f$.*

(ii) *$Y = Z$.*

(iii) *$Y \in \Pi$.*

(iv) *$U_{y,y} \subset Y$ for all y in Y.*

(v) *$U_{y,y} \subset Y$ for some y in Y.*

(vi) *$\{y\} \in \Pi$ for some y in Y.*

Proof. (a) (See the remarks following Theorem 2.3 of [22]). Suppose that (i) holds and $z \in Z^*$. Then by Theorem 5.1(b), g must equal both $\underline{f}_z - \Delta$ and $\bar{f} + \Delta$. Hence $\underline{f}_z - \bar{f} = 2\Delta = \delta$ and (ii) follows. Now, assume (ii) holds and $z \in Z^*$. Then $\bar{f} = \underline{f}_z - 2\Delta$. By Proposition 5.1 and the corollary to Theorem 5.1, we have $\|\bar{f}_z - \underline{f}_z\| = 2\Delta_z = 2\Delta$ or $\bar{f}_z \geqslant \underline{f}_z - 2\Delta = \bar{f}$. Hence, by Proposition 5.2(a), $\bar{f}_z = \bar{f}$. Now by (ii), $g = \bar{f} + \Delta = \underline{f}_z - \Delta$ and hence $g = \bar{f}_z + \Delta = \underline{f}_z - \Delta$. It follows from Proposition 5.1 that g is the unique best approximation from K_z (resp. K'_z). Thus (iii) holds. Now assume that (iii) holds and $z \in Z^*$. Since a best approximation from each K_z (resp. K'_z) is unique, by Proposition 5.1 and the corollary to Theorem 5.1, $\underline{f}_z - \bar{f}_z =$

$2\Delta_z = 2\Delta$. We conclude that $\underline{f}_z - \bar{f} = 2\Delta$ or $\bar{f} + \Delta = \underline{f}_z - \Delta$. It follows by Theorem 5.1(b) that g is unique. Thus (i) holds.

(b) For convenience let $P = \mathrm{cl}(Z)$. By Proposition 2.1, $P \in \Pi$. If g is unique and $Z^* \backslash P \neq \phi$ then let $z \in Z^* \backslash P$. By (5.1), $\bar{f}_z(z) = \theta$. Since P is closed, by hypothesis, there exists Q in Π such that Q is open in S and $z \in S \backslash Q$. Then $S \backslash Q$ is compact. Since $Y \subset Z \subset Q$ and f is continuous, we have $f(s) > \theta$ for s in $S \backslash Q$ and $f^0(Q) > \theta$. Since $z \in S \backslash Q$, we have by (4.2), $\bar{f}(z) \geqslant f^0(Q) > \theta = \bar{f}_z(z)$. This is a contradiction to $\bar{f}_z = \bar{f}$ shown in (a). Hence $P = Z^*$. Now, by (a, iii), we find that g is the unique best approximation from K_z (resp. K'_z).

Conversely suppose that $P = Z^*$. By Proposition 5.2(a), we have $\bar{f}_z = \bar{f}$ for z in Z. Now Proposition 5.2(b) shows that $\underline{f}_z$ is a continuous function of z; hence, we have $\bar{f}_z = \bar{f}$ for all z in $P = Z^*$. It follows by (a, iii) that g is unique.

(c) We denote $U_{s,s}$ by U_s for convenience. Assume (i) holds. If g is unique then $\Delta = 0$, and by (a), $\underline{f}_z - \bar{f} = 2\Delta = 0$ for all z in Z^*. By Lemma 5.3, $\bar{f}(z) = \theta$ if $z \in Z$. Hence, $\underline{f}_z(z) = \bar{f}(z) = \theta$ for all z in Z. It follows by (5.2) that $f(s) = \theta$ for all s in U_z. Hence $U_z \subset Y$ for all z in Z. Since $z \in U_z$, we have $Z \subset Y$ and hence $Z = Y$. Thus (ii) holds. Since $Z \in \Pi$, (ii) implies (iii). If (iii) holds, then by the definition of U_y, we conclude that $U_y \subset Y$ which is (iv). Clearly (iv) implies (v). If (v) holds, then by (5.2) and Lemma 5.3 we obtain $\underline{f}_y(y) = \theta = \bar{f}(y)$. Hence if g is unique then $\underline{f}_y - \bar{f} = 2\Delta$ which gives $\Delta = 0$. Thus $f \in K$ (resp. K') and this implies (i). If (vi) holds, then $U_y = \{y\} \in \Pi$ which is (v).

The proof is complete.

We remark that the condition stated in Theorem 5.2(b) is implied by D4. To see this let s and P be as stated in that condition. Then $d(s, P) > 0$. By D4 and Lemma 3.2, $Q = P_2(r)$ is in Π for sufficiently small r with $0 < r < d(s, P)$. This is the required Q. We now present some examples.

EXAMPLE 5.1. Approximation by continuous functions with level sets which are rings.

Let $X = R^n$ with any norm $|\cdot|$ and $S = \{s \in X: |s| \leqslant r\}$ with $r > 0$. Let Π consist of ϕ, S, and all rings, i.e., sets of the form $\{s \in X: \lambda \leqslant |s| \leqslant \mu\}$, $\{s \in X: \lambda < |s| < \mu\}$, $\{s \in X: \lambda \leqslant |s| < \mu\}$, and $\{s \in X: \lambda < |s| \leqslant \mu\}$, where $0 \leqslant \lambda \leqslant \mu \leqslant r$. It is easy to verify that Π satisfies conditions C1, C2, D3, and D4. Hence, by Lemma 4.1(a), K is a closed cone and, by Lemma 3.1, D1 and D2 hold. We apply the results of this section to the problem. Let $c = \min\{|y|: y \in Y\}$ and $d = \max\{|y|: y \in Y\}$. Then $Z = \{s \in X: c \leqslant |s| \leqslant d\}$. Let also c^* (resp. d^*) denote the minimum (resp. maximum) value of u (resp. v) so that $\{s \in X: u \leqslant |s| \leqslant v\} \subset Y^*$. Then $Z^* = \{s \in X: c^* \leqslant |s| \leqslant d^*\}$.

EXAMPLE 5.2. Approximation by continuous quasi-convex functions on a real interval.

Let $S=[a, b]$ be a compact real interval. Let Π consist of all the sub-intervals of S including ϕ and S. Note that a subset of S is convex if and only if it is a sub-interval of S. A function k in C is called quasi-convex if $k(\lambda s+(1-\lambda) t) \leqslant \max\{k(s), k(t)\}$ for all s, t in S and all $0 \leqslant \lambda \leqslant 1$. Equivalently, k in C is quasi-convex if and only if $\{k \leqslant \alpha\} \in \Pi$ for all real α [14]. The problem of approximation by bounded quasi-convex functions was analyzed in [18, 19] by methods of isotone optimization and sufficient conditions for a best approximation were obtained. In this section we derive from Theorems 5.1 and 5.2 stronger results including a characterization of a best approximation when f is continuous. This problem was considered recently in [24]. The quasi-convex problem on a compact convex $S \subset R^n$ was considered in [22].

It is easy to verify that conditions C1, C2, D3, and D4 hold for Π and, hence, the results of this section are applicable. Let $f \in C$ and $\theta=\min\{f(s)\}$. Let $c=\min\{s: f(s)=\theta\}$ and $d=\max\{s: f(s)=\theta\}$. Let also c^* (resp. d^*) be the minimum (resp. maximum) value of u (resp. v) such that $f(s) \leqslant \theta+2\Delta$ for all s in $[u, v]$. We then have $Z=[c, d]$ and $Z^*=[c^*, d^*]$. Clearly, if $x \in S$, then $U_{x, s}=[s, x]$ if $s<x$ and $=[x, s]$ if $s \geqslant x$, and $V_{x, s}=(s, b]$ if $s<x$ and $=[a, s)$ if $s>x$. We then obtain

$$
\begin{aligned}
\bar{f}_x(s) & =\min\{f(t): t \in[a, s]\}, && s<x, \\
& =\min\{f(t): t \in[s, b]\}, && s>x, \\
& =\theta, && s=x, \\
\underline{f}_x(s) & =\max\{f(t): t \in[s, x]\}, && s<x, \\
& =\max\{f(t): t \in[x, s]\}, && s \geqslant x .
\end{aligned}
$$

Also, $\bar{f}=\bar{f}_c$. We may now apply Theorem 5.1 to the problem. Clearly $\{s\} \in \Pi$ for all s in S. Hence, by Theorem 5.2(c), f has a unique best approximation if and only if $f \in K$. The results in [18, Sect. 4] give us, for f in C,

$$
\begin{aligned}
\Delta=\Delta_z= & (1/2) \max\{\max\{f(t)-f(s): a \leqslant s \leqslant t \leqslant z\}, \\
& \max\{f(s)-f(t): z \leqslant s \leqslant t \leqslant b\}\},
\end{aligned}
$$

for all z in Z^*. In connection with Proposition 5.1(b), we may show that $|\underline{f}_x(s)-\underline{f}_x(t)| \leqslant \omega(f,|s-t|)$ and $\|\underline{f}_x-\underline{f}_y\| \leqslant \omega(f,|x-y|)$, where $\omega(f, \delta)$ is the modulus of continuity of f.

For $O(n)$ algorithms to compute best discrete approximations for this problem under least squares and uniform norm see [20] and other references given there. In the next two examples Π is a chain of sets ordered by inclusion; i.e., if $P, Q \in \Pi$, then $P \subset Q$ or $Q \subset P$.

EXAMPLE 5.3. Approximation by continuous functions with square level sets.

This example illustrates Theorem 5.2 and shows that Z is not closed in general, although Z^* is. Let $X = R^2$ with norm $|s| = \max\{|\alpha_1|, |\alpha_2|\}$ where $s = (\alpha_1, \alpha_2) \in X$. Let $S = \bar{D}(0, 3)$ and define $E_r = D(0, r)$, $\bar{E}_r = \bar{D}(0, r)$, and $F_r = E_r \cup \{(0, r)\}$, where $r \geqslant 0$. Let Π consist of the sets E_r, $\bar{E}_r$, and F_r for $0 \leqslant r \leqslant 3$. Clearly, C1, C3, D3, and D4 hold. By D4, the condition stated in Theorem 5.2(b) may also be shown to hold. By Lemma 4.1(b), K is a closed convex cone.

Let $s_0 = (0, 1)$ and s_i, $1 \leqslant i \leqslant 4$, denote the four corner points of $\bar{D}(0, 1)$. Let f denote the greatest convex minorant (gcm) of the following five points in R^3: $(s_0, 0)$ and $(3s_i, 3)$, $1 \leqslant i \leqslant 4$. In other words, f is the largest convex function on S whose graph lies no higher than these five points in R^3. It is easy to see that $\bar{f}$ is the gcm of the eight points $(s_i, 0)$ and $(3s_i, 3)$, $1 \leqslant i \leqslant 4$. The following may be easily verified: $\Delta = \frac{3}{4}$; $Y = \{s_0\}$; $Z = F_1$; $\mathrm{cl}(Z) = Z^* = \bar{E}_1$; Y^* is the convex hull of $(\frac{3}{2}, 2)$, $(-\frac{3}{2}, 2)$, $(-\frac{3}{2}, -1)$, and $(\frac{3}{2}, -1)$; and $\bar{f} + \Delta$ is the unique best approximation. Thus Z is not closed but Z^* is. Again, statement (ii) of Theorem 5.2(c) does not apply and we have verified that f defined above, which is not in K, has a unique best approximation. Now define f as follows: On $\bar{E}_2$, f equals the gcm of $(s_i, 0)$ and $(2s_i, 3)$, $1 \leqslant i \leqslant 4$; on $S \backslash E_2$ it equals the smallest concave majorant of $(2s_i, 3)$ and $(3s_i, 2)$, $1 \leqslant i \leqslant 4$. Then f is in $C \backslash K$ and does not have a unique best approximation, as may be easily seen. This, again, verifies Theorem 5.2(c) since $Y = \bar{E}_1 \in \Pi$ which is statement (ii) of that theorem.

EXAMPLE 5.4. Approximation by continuous non-decreasing functions.

Let $S = [0, 1]$ be a real interval and Π consist of sets of the form $[0, c)$, $[0, c]$, where $0 \leqslant c \leqslant 1$. Then C1, C3, D3, and D4 hold and K is the closed convex cone of non-decreasing functions. Let $f(s) = -2s$ on S. Then $g(s) = -1$ is the unique best approximation, $\Delta = 1$, $Y = \{1\}$, $\mathrm{cl}(Z) = Z = Z^* = Y^* = S$. This example shows that statement (ii) of Theorem 5.2(c) does not apply, and, hence, uniqueness holds even when f is not in K. This example is considered in the next section in more detail.

6. BEST APPROXIMATION FROM A CONVEX CONE K

In this section, we identify extremal best approximations and OLSOs when Π satisfies C1, C3, D1, and D2. By Lemma 4.1(b), K is then a closed convex cone. For s in S, let

$$U_s = \bigcap \{P \in \Pi : s \in P\},$$

$$V_s = \bigcup \{P \in \Pi : s \in S \backslash P\}.$$

Condition C1 (resp. C3) implies that U_s (resp. V_s) is in Π.

PROPOSITION 6.1. *Assume* $C1$, $C3$, $D1$, *and* $D2$ *hold. Let* $f \in C$ *and define*

$$\bar{f}(s) = \inf\{f(t) : t \in S \setminus V_s\}, \qquad s \in S,$$
$$\underline{f}(s) = \sup\{f(t) : t \in U_s\}, \qquad s \in S.$$

Then $\bar{f} \in C$ *and is the greatest* K- *(or* K'-*) minorant of* f; $\underline{f} \in C$ *and is the smallest* K- *(or* K'-*) majorant of* f. *(Both* $\bar{f}$ *and* $\underline{f}$ *may be characterized by statements similar to Proposition* 4.1(b.)

Proof. It has been shown in Proposition 4.4 of [22] that $\bar{f}$ and $\underline{f}$ are respectively the greatest K'-minorant and the smallest K'-majorant of f in B. To prove the assertions it suffices to show that $\bar{f}$ and $\underline{f}$ are in C when f is in C. This follows from conditions D1 and D2 as in the proof of continuity of $\bar{f}$ in Proposition 4.1. The proof is complete.

Clearly $U_s = \bigcap \{U_{x,s} : x \in S\} = U_{s,s}$ and $V_s = \bigcup \{V_{x,s} : x \in S\}$, where $U_{x,s}$ and $V_{x,s}$ are defined in Section 5. Hence, by (5.1) and (5.2), we have for all s in S

$$\bar{f}(s) = \sup\{\bar{f}_x(s) : x \in S\},$$
$$\underline{f}(s) = \inf\{\underline{f}_x(s) : x \in S\}.$$

THEOREM 6.1. Characterization of best approximations and uniqueness. *Assume* $C1$, $C3$, $D1$, *and* $D2$ *hold. Let* $f \in C$. *Then*

$$\Delta'(f) = \Delta(f) = (1/2)\,\|f - \bar{f}\| = (1/2)\,\|f - \underline{f}\| = (1/2)\,\|\underline{f} - \bar{f}\|.$$

Also $\bar{f} + \Delta(f)$ *and* $\underline{f} - \Delta(f)$ *are in* C *with* $\underline{f} - \Delta(f) \leqslant \bar{f} + \Delta(f)$ *and are, respectively, the maximal and minimal best approximation to* f *from* K *(or* K'*). Furthermore, a* g *in* K *(or* K'*) is a best approximation to* f *if and only if* $\underline{f} - \Delta(f) \leqslant g \leqslant \bar{f} + \Delta(f)$. *If* $f' = (\bar{f} + \underline{f})/2$, *then the operator* $T: C \to K$ *defined by* $T(f) = f'$ *is the unique OLSO with* $c(T) = 1$. *A best approximation* g *is unique if and only if* $\underline{f} - \bar{f} = \delta$ *for some* $\delta \geqslant 0$, *and in this case* $\delta = 2\Delta(f)$ *and* $g = \bar{f} + \Delta(f) = \underline{f} - \Delta(f)$.

Proof. By Lemma 4.1(c) and Proposition 6.1, K (or K') satisfies all three conditions stated in Section 1 of [22]. The result follows from Theorem 2.1 of [22]. The uniqueness statement is established in the remarks following Theorem 2.3 of [22] and uses the characterization of a best approximation derived there. The proof is complete.

We now remark on uniqueness. Note that the results of Section 5 are applicable to this problem and, hence, Theorem 5.2 holds. Note that Π satisfies C1 and C3 and, hence, so does Π'. It is easy to see that under C1 and C3, $-K$ (resp. $-K'$) is the set of all k in C (resp. B) such that $\{k \leqslant \alpha\} \in \Pi'$ for all α. Hence, the problem of finding a best approximation to f form K (resp. K') is equivalent to the symmetric problem of finding a best approximation to $-f$ from $-K$ (resp. $-K'$). This observation allows us to obtain results analogous to Theorem 5.2 as follows. We defined sets K_x, K'_x, $U_{x,s}$, Y, Z, Y^*, and Z^* in Section 5. Analogously, we may define sets M_x, M'_x, $W_{x,s}$, G, H, G^*, and H^* by replacing f, Π, Π_s, P, Y, Z, Y^*, and Z^*, respectively, by $-f$, Π', Π'_s, P', G, H, G^*, and H^* in their definitions. Then results symmetric to the uniqueness Theorem 5.2 may be obtained by replacing the mathematical entities there by their symmetric analogs and the condition stated in Theorem 5.2(b) by its symmetric version. We now present some examples particularly to illustrate uniqueness. In what follows let Δ denote $\Delta(f)$. The proof of the following lemma is straightforward.

LEMMA 6.1. *$0 \leqslant \Delta \leqslant (\mu - \theta)/2$ where $\mu = \max(f)$ and $\theta = \min(f)$. There exists a best approximation g which is identically equal to a constant if and only if $\Delta = (\mu - \theta)/2$ and in this case $g = (\mu + \theta)/2$.*

EXAMPLE 6.1. Approximation by continuous non-decreasing functions. Let $S = [a, b]$, a compact real interval, and Π consist of all intervals of the form $[a, s)$ and $[a, s]$, where $a \leqslant s \leqslant b$. Then C1, C3, D3, and D4 hold and by Lemma 3.1, D1 and D2 apply. Clearly, K is the closed convex cone of non-decreasing functions and Theorem 6.1 holds. We investigate uniqueness in the following proposition. Let μ and θ be as defined in Lemma 6.1.

PROPOSITION 6.2. *Let $f \in C \backslash K$ and g be a best approximation to f. Then the following* (a) *and* (b) *are equivalent and imply* (c):

(a) *g is unique.*

(b) *$f(a) = \mu$ and $f(b) = \theta$.*

(c) *g identically equals $(\mu + \theta)/2$ and $\Delta = (\mu - \theta)/2$.*

Proof. Suppose (a) holds; we establish (b). Let $\delta = 2\Delta > 0$. We assert that $\mu - \theta = \delta$. Since $f \in C \backslash K$ and g is unique, by Theorem 6.1, we have $\underline{f} - \bar{f} = \delta = 2\Delta > 0$. Clearly, $U_s = [a, s]$ and $V_s = [a, s)$. Using the definition of $\underline{f}$ and $\bar{f}$, we obtain from $\underline{f}(s) - \bar{f}(s) = \delta$ the following:

$$\max\{f(t) : t \in [a, s]\} - \min\{f(t) : t \in [s, b]\} = \delta, \qquad s \in [a, b]. \quad (6.1)$$

For convenience, let $f(a) = \alpha$ and $f(b) = \beta$. Then with $s = a$ and b in (6.1) we obtain $\alpha - \theta = \delta$ and $\mu - \beta = \delta$. Hence, $\alpha > \theta$ and $\beta < \mu$. Let y (resp. z) denote a point at which the minimum (resp. maximum) of f is attained. Then $a < y$ and $z < b$. Suppose first that there exists a pair y, z with $z < y$.

Then (6.1) with $s=z$ gives $\mu-\theta=\delta$. Hence, $\alpha=\mu$ and $\beta=\theta$ and (b) holds. Also, Lemma 6.1 shows that $\mu-\theta=\delta=2\Delta$ implies (c). Now assume that for every pair y, z we have $y<z$. This assumption implies that $\alpha=f(a)<\mu$ and $\beta=f(b)>\theta$. We shall reach a contradiction. Let y and z with $y<z$ denote the largest such y and smallest such z. Then $\theta<f(s)<\mu$ for all s in (y, z). Now (6.1) with $s=y$ and z, respectively, shows that $f(s)\leqslant\theta+\delta=\alpha$ for all s in $[a, y]$ and $f(s)\geqslant\mu-\delta=\beta$ for all s in $[z, b]$. Let v be the largest point in $[y, z]$ such that $f(v)=\beta$. Since $\beta<\mu$ we have $v<z$. Again since $f(s)\geqslant\beta$ for s in $[v, b]$ and $f(b)=\beta$, we obtain from (6.1) with $s=v$ that $\max\{f(t): t\in[a, v]\}=\mu$. Since $f(s)<\mu$ for s in $[y, v]$ and $f(s)\leqslant\alpha<\mu$ for all s in $[a, y]$, we have reached a contradiction. It is obvious that (b) implies (a). The proof is complete.

To show that (c) does not necessarily imply (a) in the above proposition consider the following example: Let $S=[0, 3]$ and define f by $f(0)=f(2)=-1$, $f(1)=f(3)=1$ and by linear interpolation everywhere else. Then $\Delta=1$, and $\underline{f}-\Delta$ and $\bar{f}+\Delta$, as stated in Theorem 6.1, are two distinct best approximations.

Example 6.2. Approximation by continuous functions with level sets which are balanced intervals.

Let $S=[-b, b]$ where $b>0$ and Π consist of all intervals of the form $(-s, s)$ and $[-s, s]$ where $0\leqslant s\leqslant b$. Then clearly Π satisfies C1, C3, D3, and D4. Thus Theorem 6.1 holds.

Proposition 6.3. *Let $f\in C\backslash K$. Suppose there exist some y, z with $|y|\geqslant|z|$, where $f(y)=\theta$ and $f(z)=\mu$. If a best approximation is unique then it identically equals $(\mu+\theta)/2$ and $\Delta=(\mu-\theta)/2$.*

Proof. Clearly $U_s=[-|s|, |s|]$ and $V_s=(-|s|, |s|)$. Then as in the previous example, $\underline{f}-\bar{f}=\delta=2\Delta>0$ gives

$$\sup\{f(t): t\in[-s, s]\}-\inf\{f(t): t\in[-b, -s]\cup[s, b]\}=\delta, \qquad 0\leqslant s\leqslant b.$$

Letting $s=|y|$ in the above equation, we obtain $\mu-\theta=\delta$. Then the conclusion follows by Lemma 6.1. The proof is complete.

To show that the condition $|y|\geqslant|z|$ in the above proposition cannot be dropped, consider the following example: $b=2$, f defined by $f(-2)=1$, $f(-1)=0$, $f(0)=f(1)=2$, $f(2)=3$, and by linear interpolation everywhere else. Then g defined by $g(-2)=g(2)=2$, $g(-1)=g(1)=1$, and by linear interpolation everywhere else, is the unique best approximation.

Example 6.3. Approximation by continuous functions with polyhedral level sets.

Let $X=R^n$ with Euclidean norm and $S\subset R^n$ be compact convex and polyhedral. A polyhedral set is defined to be the intersection of finitely many closed half spaces; such a set is necessarily convex. Let A be an $m \times n$ matrix and $b: R \to R^m$ be a vector-valued continuous function satisfying the following for each $i: b_i(\lambda) < b_i(\mu)$ if $\lambda < \mu$ and $b_i(\lambda) \to \pm\infty$ as $\lambda \to \pm\infty$ where $b = (b_1, b_2, ..., b_m)$. Let Π consist of all the sets of the form $\{s \in S: As \leqslant b(\lambda)\}$ and $\{s \in S: As < b(\lambda)\}$, where $\lambda \in R$. Clearly, the closed sets in Π are polyhedral and ϕ, $S \in \Pi$. It is easy to verify that C1, C3, D1, and D2 apply. Hence K is a closed convex cone and Theorem 6.1 holds.

EXAMPLE 6.4. Approximation by continuous functions with level sets which are intervals containing a given point.

We revert to Example 4.1 for the star-shaped case and observe that both Ω and Π in that example satisfy C1 and C3, but the results are weaker than those in Theorem 6.1 above because D4 or D2 does not apply to Π. To show that the results in that example cannot be strengthened, consider its special case when $S = [0, 2]$ and $x = 1$. Then Π consists of all star-shaped subsets of S relative to x which are intervals containing x, including ϕ and S. Let f on S be given by $f(s) = s$ on $[0, 1]$ and 1 on $[1, 2]$. Then $\bar{f}(s) = 0$ on $[0, 1]$, $\bar{f}(s) = 1$ on $(1, 2]$, and $\underline{f}(s) = 1$ on S. Thus $\bar{f}$ is not continuous and the maximal best approximation does not exist.

7. APPLICATIONS TO NORMED VECTOR LATTICES

In this section, we derive several results for normed vector lattices. We show that the set transformation developed in Section 3 can be applied to lower and upper subsets of an order-interval in a lattice. We also investigate the compactness of order-intervals. We consider an approximation problem on a lattice and apply results of Section 6 to characterize best approximations. Throughout this section we assume that $S \subset X$ is not necessarily compact unless otherwise stated.

A normed (vector) lattice X is a vector lattice (Riesz space) equipped with a norm $|\cdot|$. The axioms which a normed lattice satisfies are given in [16, 25]. We use the following notation: $\leqslant$ for the partial order on X, $s \vee t$ (resp. $s \wedge t$) for supremum (resp. infimum) of a pair of elements s, t in X, $s^+ = s \vee 0$ (resp. $s^- = (-s) \vee 0$) for the positive (resp. negative) part of s, and $s^* = s^+ + s^-$ for the absolute value of s in X. The norm $|\cdot|$ has the property that $|s| \leqslant |t|$ whenever $s, t \in X$ and $s^* \leqslant t^*$. For s, t in X, we sometimes write $t \geqslant s$ in place of $s \leqslant t$; also $s < t$ (resp. $s > t$) means $s \leqslant t$ (resp. $s \geqslant t$) but $s \neq t$.

A subset P of S is called a lower (resp. upper) subset of S if $s \in P$ and $t \in S$ with $t \leqslant s$ (resp. $t \geqslant s$) then $t \in P$. Clearly, P is a lower (resp. upper) subset of S if and only if $S \backslash P$ is an upper (resp. lower) subset of S. Also, ϕ and S are both lower and upper subsets of S. Note that the set of all lower (resp. upper) subsets of S is closed under arbitrary intersections and unions; hence, it satisfies conditions C1 and C3. If $a, b \in X$ and $a \leqslant b$, then $[a, b] = \{s \in X: a \leqslant s \leqslant b\}$ is called an order-interval of X. It is easy to verify that an order-interval is closed and convex. A real function f on $S \subset X$ is called isotone (resp. antitone) if $f(s) \leqslant f(t)$ (resp. $f(s) \geqslant f(t)$) whenever $s, t \in S$ and $s \leqslant t$. The proof of the following lemma is simple; equivalence of (b) and (c) follows because conditions C1 and C3 apply to lower and upper sets (Lemma 4.1(d)).

LEMMA 7.1. *Let f be a real function defined on $S \subset X$. Then the following are equivalent.*

(a) *f is isotone (resp. antitone) on S.*

(b) *$\{f \leqslant \alpha\}$ is a lower (resp. upper) subset of S for all real α.*

(c) *$\{f < \alpha\}$ is a lower (resp. upper) subset of S for all real α.*

The following lemma is fundamental in establishing certain results in this section.

LEMMA 7.2. *Let $S = [a, b]$ be an order-interval and $s, t \in S$ with $s \leqslant t$. If $u, v \in S$ then there exist $x, y \in S$ with $x \geqslant u$, $y \leqslant v$ and $|x - t| \leqslant |u - s|$, $|y - s| \leqslant |v - t|$.*

Proof. If $z = v - (t - s)$ then $z \leqslant v \leqslant b$ and $|z - s| = |v - t|$. Let $y = z \vee a$. Then $a \leqslant y \leqslant v \leqslant b$ and hence $y \in S$. Now $y - s = (z - s) \vee (a - s)$. Since $a - s \leqslant 0$, we have $(y - s)^+ = (z - s)^+$ and $(y - s)^- = (z - s)^- \wedge (a - s)^- \leqslant (z - s)^-$. Thus $(y - s)^* \leqslant (z - s)^*$ and hence $|y - s| \leqslant |z - s| = |v - t|$. We may prove the other case in a similar manner by letting $z = u + t - s$ and $x = z \wedge b$. The proof is complete.

If $P \subset S$, we denote by $\mathrm{int}(P)$ and $\mathrm{cl}(P)$ the interior and closure of P with respect to the relative topology for S.

PROPOSITION 7.1. *Let $S = [a, b]$ be an order-interval and $P \subset S$. If P is a lower (resp. upper) set, then* $\mathrm{int}(P)$ *and* $\mathrm{cl}(P)$ *are lower (resp. upper) sets.*

Proof. To show $\mathrm{int}(P)$ is a lower set when P is lower, let $s \in \mathrm{int}(P)$, $t \in S$, and $t \leqslant s$. There exists $\rho > 0$ such that $D(s, \rho) \cap S \subset P$. Let $v \in D(t, \rho) \cap S$. By Lemma 7.2, there exists $u \in S$ such that $v \leqslant u$ and $|u - s| \leqslant |v - t| < \rho$. It follows that $u \in D(s, \rho) \cap S \subset P$. Now since P is lower, $v \in P$. Again, since v is arbitrary, we have $D(t, \rho) \cap S \subset P$. Thus $t \in \mathrm{int}(P)$ and $\mathrm{int}(P)$ is lower. Similarly we may show that $\mathrm{cl}(P)$ is lower when P is lower. Since P is upper if and only if $S \backslash P$ is lower, the results for upper sets follow from those for lower sets. The proof is complete.

Recall the set transformations $P_i(r)$ and $\bar{P}_i(r)$, $i=1, 2$, of $P \subset S$ defined in Section 3.

THEOREM 7.1. *Let $S=[a, b]$ be an order-interval.*

(a) *Let $P \subset S$ be a non-empty lower (resp. upper) set. Let m denote $\sup\{d(s, S \setminus P) : s \in P\}$. Then $d(s, S\setminus P)$ is antitone (resp. isotone) on S. Consequently, $P_1(r)$, $r \geqslant 0$, and $\bar{P}_1(r)$, $r>0$, are lower (resp. upper) subsets of S which are contained in P with $P_1(r)=\operatorname{int}(\bar{P}_1(r))$ for $r>0$. ($\bar{P}_1(0)=S$ which is both lower and upper.) If $\operatorname{int}(P) \neq \phi$ then $m>0$, and for $r<m$, $P_1(r)$ and $\bar{P}_1(r)$ are non-empty.*

(b) *Let $P \subset S$ be a non-empty lower (resp. upper) set. Then $d(s, P)$ is isotone (resp. antitone) on S. Consequently, for all $r \geqslant 0$, $P_2(r)$ and $\bar{P}_2(r)$ are non-empty lower (resp. upper) subsets of S containing P with $\bar{P}_2(r)= \operatorname{cl}(P_2(r))$.*

Proof. We establish (a); the proof for (b) is similar. Let P be lower and $d(s)$ denote $d(s, S\setminus P)$. We show that if $s, t \in S$ and $s \leqslant t$ then $d(s) \geqslant d(t)$. If $d(t)=0$ then $d(s) \geqslant d(t)$. Hence, suppose that $d(t)>0$. Then $D(t, d(t)) \cap S \subset P$. We assert that $A=D(s, d(t)) \cap S \subset P$. If $u \in A$ then by Lemma 7.2, there exists v in S with $u \leqslant v$ and $|v-t| \leqslant |u-s| < d(t)$. Thus $v \in D(t, d(t)) \cap S \subset P$. Since P is lower, $u \in P$, and hence, $A \subset P$. By the definition of d we then have $d(s) \geqslant d(t)$. By antitonicity of d and Lemma 7.1, we conclude that $P_1(r)=\{d>r\}$ and $\bar{P}_1(r)=\{d \geqslant r\}$ are lower sets for $r \geqslant 0$. Since S is closed and convex, by Proposition 3.1(a), we have $P_1(r)=\operatorname{int}(\bar{P}_1(r))$ for $r>0$. (Note that compactness of S is not needed in that proposition for this result to hold.) For $r<m$, the sets are clearly non-empty. The proof when P is upper is similar. The proof is complete.

We remark that by Proposition 3.1 we have $\operatorname{int}(P)=P_1(0)$ and $\operatorname{cl}(P)= \bar{P}_2(0)$. Hence Proposition 7.1 also follows from Theorem 7.1. The following proposition gives a property of the nearest elements.

PROPOSITION 7.2. *Let $S=[a, b]$ be an order-interval and $P \subset S$ be a lower (resp. upper) set. If $s \in S\setminus P$ and there exists an element t in P nearest to s, then there exists an element $u \leqslant s$ (resp. $u \geqslant s$) in P nearest to s.*

Proof. Suppose that P is lower and $s \in S\setminus P$. Define $u=s \wedge t$. Then $u \leqslant s$ and $u \in S$. Also $u \leqslant t$ and hence $u \in P$ since P is lower. Now $u-s=0 \wedge (t-s)$. Hence $(u-s)^+=0$ and $(u-s)^-=(t-s)^-$. Thus $(u-s)^*=(t-s)^- \leqslant (t-s)^*$. We conclude that $|u-s| \leqslant |t-s|$ and, hence, $|u-s|=|t-s|$ since $u \in P$. An analogous proof may be given for the other case. The proof is complete.

In the rest of the section, we demonstrate the existence of a normed lattice in which every infinite dimensional order-interval is compact since such a lattice makes Example 7.2 more significant. The following simple lemma provides some insight into the example. A subset of X is called order-bounded if it is contained in some order-interval [25].

LEMMA 7.3. *If in a normed lattice every order-interval is compact and every norm-bounded subset is also order-bounded, then that lattice is finite dimensional.*

Proof. The hypothesis implies that every norm-bounded subset is relatively compact; i.e., its closure is compact. The required conclusion then follows by [7, Chap. IV, Theorem 3]. The proof is complete.

By the above lemma, our infinite dimensional lattice must not have the property stated there.

EXAMPLE 7.1. A vector lattice whose every infinite dimensional order-interval is compact.

Let $X = l_1$, which is the linear space of absolutely summable real sequences $s = (\alpha_m)$ with norm $|s| = \sum \max\{\alpha_m, -\alpha_m\} < \infty$. We define the ordering $\leqslant$ on X as follows: If $s = (\alpha_m)$ and $t = (\beta_m)$ then $s \leqslant t$ if and only if $\alpha_m \leqslant \beta_m$ for all m. It is easy to verify that X is a normed lattice and, since it is complete, it is a Banach lattice. Let e_m be the sequence with unity in the mth position and zero everywhere else. Then $e_m \in X$ and $|e_m| = 1$.

LEMMA 7.4. *Every order-interval $[a, b]$ of $X = l_1$ is compact. Furthermore, if $b - a = (\lambda_m) \geqslant 0$ and $M = \{m: \lambda_m > 0\}$, then $[a, b]$ is infinite dimensional if and only if M is infinite.*

Proof. Since $[a, b] = a + [0, b - a]$ and $b - a \geqslant 0$, it suffices to show that $J = [0, s]$ is compact, where $s = (\alpha_m) \geqslant 0$. Let $s_n = (\alpha_{n,m})$ be a sequence in J. Then $0 \leqslant \alpha_{n,m} \leqslant \alpha_m$ for all n, m. Now, the well-known Cantor diagonalization process gives a subsequence $t_n = (\beta_{n,m})$ of s_n and an element $t = (\beta_m)$ such that $\beta_{n,m} \to \beta_m$ for each m as $n \to \infty$. See, e.g., [12, p. 220]. Since $0 \leqslant t_n \leqslant s$ for all n, we find that $t \in J$ and applying the bounded convergence theorem [6] we obtain $|t_n - t| \to 0$ as $n \to \infty$. Thus J is compact. To prove the second statement of the lemma, define $u_m = a + \lambda_m e_m$. Then $u_m \in [a, b]$ for all m. Since the vectors $\{\lambda_m e_m: m \in M\}$ are linearly independent, the required conclusion immediately follows. The proof is complete.

With reference to Lemma 7.3, we remark that $E = \{e_m: m \geqslant 1\}$ gives a norm-bounded subset of X which is not order-bounded. We now consider an approximation problem on a normed lattice and apply the above results.

EXAMPLE 7.2. Approximation by continuous isotone functions on a normed lattice.

Let $S=[a,b]$ be a compact order interval in a normed lattice X. As shown in Example 7.1, infinite dimensional compact order intervals exist; such intervals make this problem more significant. Given a continuous function f on S, the problem is to find a best approximation to f from the class of all continuous isotone functions on S.

Let Π be the set of all lower subsets of S including ϕ and S. As observed before, Π satisfies C1 and C3. By Lemma 7.1, K (resp. K') is precisely the set of all continuous (resp. bounded) isotone functions on S; K (resp. K') is a closed convex cone. Now, by Theorem 7.1, Π satisfies D3 and D4, and hence, by Lemma 3.1, it satisfies D1 and D2. We conclude that Proposition 6.1 and Theorem 6.1 are applicable to K (K'). It is easy to verify that $S\backslash V_s=[s,b]$ and $U_s=[a,s]$ in Proposition 6.1 for this problem. The following uniqueness result also holds; its proof is similar to Proposition 6.3. Let $f\in C\backslash K$, and μ and θ be as in Lemma 6.1. Suppose there exist some y, z in S with $y>z$ where $f(y)=\theta$ and $f(z)=\mu$. If a best approximation is unique, then it identically equals $(\mu+\theta)/2$ and $\Delta=(\mu-\theta)/2$. The reader may easily construct an example of f to show that there exists a non-constant unique best approximation to f when the stated condition does not hold.

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